AIAA 81-0499R

Nonlinear In-Plane and Out-of-Plane Vibrations in Solar Arrays

Michail A. Zak*
Jet Propulsion Laboratory, California Institute of Technology, Pasadena, Calif.

This paper presents the results of analytical investigations of nonlinear effects in in-plane and out-of-plane vibrations of solar arrays caused by surface wrinkling of solar array blankets. It is shown that nonlinearities generated by wrinkles are important even for small deformations. An approximate analytical solution describing the frequency dependencies of the amplitudes of in-plane vibrations is derived. A parametrical resonance of out-of-plane vibrations induced by in-plane vibrations is described.

Introduction

THE work described here investigates the nonlinear effects in in-plane and out-of-plane vibrations of solar arrays caused by surface wrinkling of solar array blankets. From the point of view of structural analysis, a solar array is considered to be a thin film stretched between a drum and an elastic bistem boom (Fig. 1). Clearly, in-plane shear and out-of-plane vibrations of the film are generated by bending and torsion of the bistem boom, respectively.

Test data for the shear force/deflection characteristic[1] (Fig. 2) have shown that a jump of in-plane shear stiffness emerges in the course of very small deformations where geometrical and physical nonlinearities are usually negligible. The proper explanation of such a phenomenon with corresponding dynamical effects cannot be given within the frames of conventional models of film because these models allow compression, while in reality the thin film does not allow such compression but rather forms wrinkles in the directions orthogonal to the principal direction of compression. The nonlinear behavior of wrinkles explains the above-mentioned phenomenon. Some statics and dynamics problems of wrinkling have been considered previously.[2-5] The dynamical behavior of a single wrinkle was also considered.[6] This paper presents a theoretical explanation of the nonlinear force-deflection characteristic mentioned above, with approximate analytical investigations of the corresponding dynamical effects.

Shearing Stiffness of a Stretched Film

Let us consider an elastic film forming a blanket of a solar array which is stretched between two parallel supports AB and CD as shown in Fig. 3. Assume that the support AB can be shifted horizontally by small in-plane bending of the boom OB while the vertical deflection AB is negligible. After a small horizontal shift δ, the strains in film are given by the formulas

$$\epsilon_{xx} = -\nu\frac{T_0}{E} \quad \epsilon_{yy} = \frac{T_0}{E} \quad \epsilon_{xy} = \frac{\alpha}{2} \tag{1}$$

where T_0 is the pretension in the vertical direction y, E the elastic modulus of the film (together with the springs of the tensionings systems), ν the Poisson's ratio of the film, and α the shear angle

$$\alpha = \delta/\ell < 0.1 \tag{2}$$

where $\ell = AC$ is the length of the film. Because all the strains are assumed to be small there is no difference between their components in the initial and new systems of coordinates.

The principal strains ϵ_{11}, ϵ_{22}, and angle β between the vertical axis y and the direction of the maximum positive strains ϵ_{11} are

$$\epsilon_{11},\ \epsilon_{22} = \tfrac{1}{2}(\tilde{T}_0^1 \pm \sqrt{\tilde{T}_0^2 + \alpha^2}) \tag{3}$$

$$\beta = \operatorname{arctg}\frac{\sqrt{\tilde{T}_0^2 + \alpha^2} - \tilde{T}_0}{\alpha} \tag{4}$$

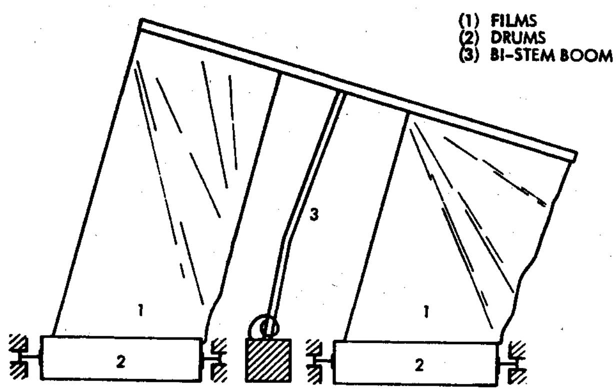

Fig. 1 Structural model of solar array.

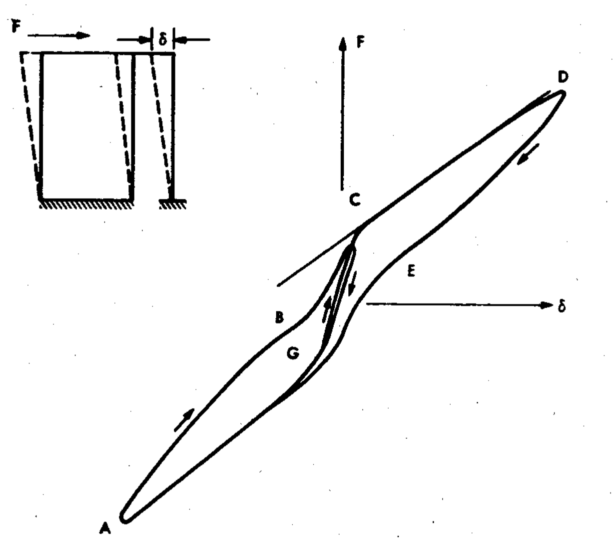

Fig. 2 In-plane force-deflection characteristic for solar array.

Received Jan. 28, 1981; presented as Paper 81-0499 at the AIAA/ASME/ASCE/AHS 22nd Structures, Structural Dynamics and Materials Conference, Atlanta, Ga., April 6-8, 1981; revision received July 21, 1981.

*Member Technical Staff, Applied Mechanics Technology Section.

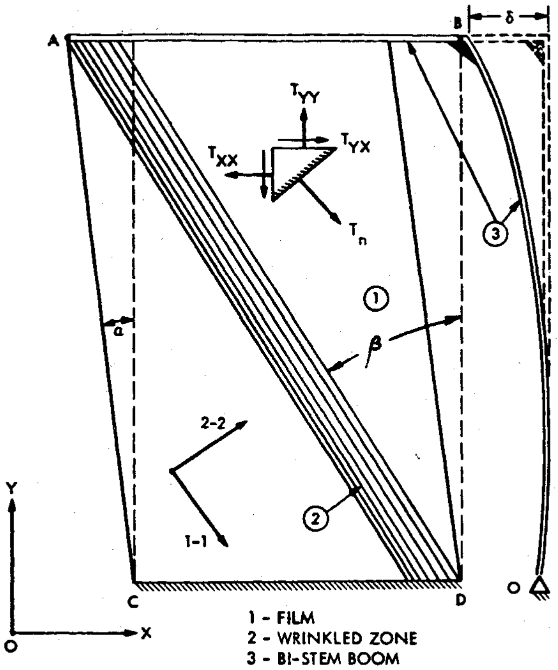

Fig. 3 Stresses in the shearing film.

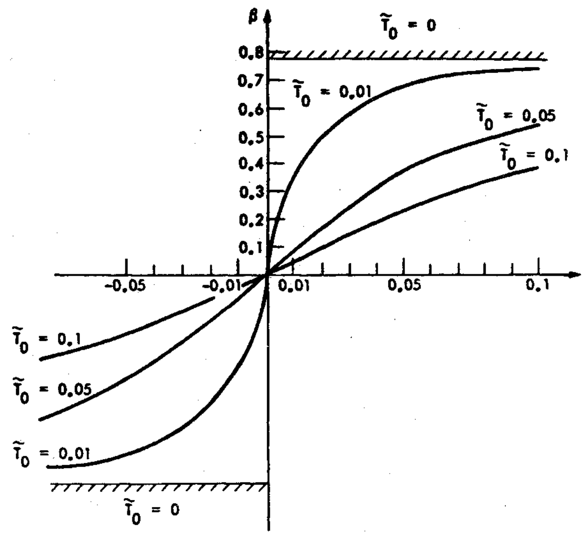

Fig. 4 Dependence $\beta=f(\alpha,\ \tilde{T}_0)$.

where

$$\tilde{T}_0=\frac{T_0(1+\nu)}{E},\quad \tilde{T}_0^1=\frac{T_0}{E}(1-\nu) \tag{5}$$

$$\epsilon_{11}<0,\quad \epsilon_{22}<0 \tag{6}$$

which means that the film is elongated in the principal stress direction 1-1 and contracted in the principal stress direction 2-2. In such a situation[5] the principal stresses are defined by the following equations:

$$T_{11}=E\epsilon_{11},\quad T_{22}=0 \tag{7}$$

where

$$T_{xx}=\frac{E}{2}(\tilde{T}_0^1+\sqrt{\tilde{T}_0^2+\alpha^2})\sin^2\beta \tag{8}$$

$$T_{yy}=\frac{E}{2}(\tilde{T}_0^1+\sqrt{\tilde{T}_0^2+\alpha^2})\cos^2\beta \tag{9}$$

$$T_{xy}=\frac{E}{4}(\tilde{T}_0^1+\sqrt{\tilde{T}_0^2+\alpha^2})\sin^2\beta \tag{10}$$

Now let us analyze more carefully the dependence between the principal stress direction 1-1 (the direction of wrinkles) and the shear angle which is given by Eq. (4).

First of all, it is easy to show that the angle β can be sufficiently large even for very small α. Indeed

$$\lim_{\alpha\to 0}\beta=\lim_{\alpha\to 0}\ \mathrm{arc}\ tg\frac{\sqrt{\tilde{T}_0^2+\alpha^2}-\tilde{T}_0}{\alpha} \tag{11}$$

$$=\begin{cases}0 & \text{if } \tilde{T}_0\neq 0\\ \pi/4\cdot\mathrm{sign}\alpha & \text{if } \tilde{T}_0=0\end{cases}$$

Obviously, for a film without pretension, the angle β does not depend on a shearing deformation at all for $\alpha\neq 0$, but has a jump from $(-\pi/4)$ to $(+\pi/4)$ when α changes sign. This phenomenon can be observed: the directions of wrinkles become unstable, jumping from $(-\pi/4)$ to $(+\pi/4)$ when the shear angle changes sign. The function $\beta=f(\alpha,\tilde{T}_0)$ is plotted for different $\tilde{T}_0$ in Fig. 4. Clearly, β has a strong nonlinearity at the point $\alpha=0$ even for $\tilde{T}_0\neq 0$. It must be emphasized that this nonlinearity is important even for very small shear angles where geometrical and physical nonlinearities are negligible.

As shown in Ref. 5, wrinkles form a family of straight lines if there are no forces applied to the film and the tension along any wrinkle is constant. Hence, a wrinkle cannot intersect the free edges of the film where

$$T_{xx}=0,\quad T_{xy}=0,\quad \epsilon_{22}=0 \tag{12}$$

while the wrinkle emerges if $\epsilon_{22}<0$. Thus the width of the stretched area is

$$b^*=b[1-(\ell/b)(tg\beta-\alpha)+|1-(\ell/b)(tg\beta-\alpha)|]/2\geq 0 \tag{13}$$

where $b=AB$ is the width of the film (Fig. 3). Clearly, this effective width b^* vanishes if

$$tg\beta-tg\alpha\geq b/\ell \tag{14}$$

The equality in Eq. (14) is valid, for instance, when

$$\tilde{T}_0=0,\qquad \alpha=1-(b/\ell) \tag{15}$$

The function $b^*/b=\phi(\alpha,\tilde{T}_0,b/\ell)$ is plotted for $T_0=0$, 0.1, and different ratios b/ℓ in Fig. 5.

Now the total shearing force is expressed via the shearing stresses T_{xy} and effective width b^*

$$F=\frac{E}{4}[(\tilde{T}_0'+\sqrt{\tilde{T}_0^2+\alpha^2})\sin^2\beta]b^* \tag{16}$$

where β and b^* are defined by Eqs. (4) and (13), respectively. Such an in-plane force-deflection characteristic is plotted in Fig. 6 for $\tilde{T}_0=0.05$, $b/\ell=0.2$; $\tilde{T}_0=0$, $b/\ell=0.95$; and $\tilde{T}_0=0.1$, $b/\ell=0.22$.

Thus even small variations of prestress $\tilde{T}_0$ or the ratio b/ℓ can completely change the type of nonlinearities of that characteristic. As shown in Fig. 6, there are three different types of nonlinearities which can be reduced approximately to the standard nonlinearities (see dashed lines): 1) the nonlinearity with a zone of insensitivity; 2) the nonlinearity with a zone of saturation; and 3) the impulse nonlinearity. Now the theoretical analog of the experimental characteristic shown in Fig. 1 is obtained by adding the effective spring force $F_\beta=C\alpha$ of the boom in the course of the in-plane

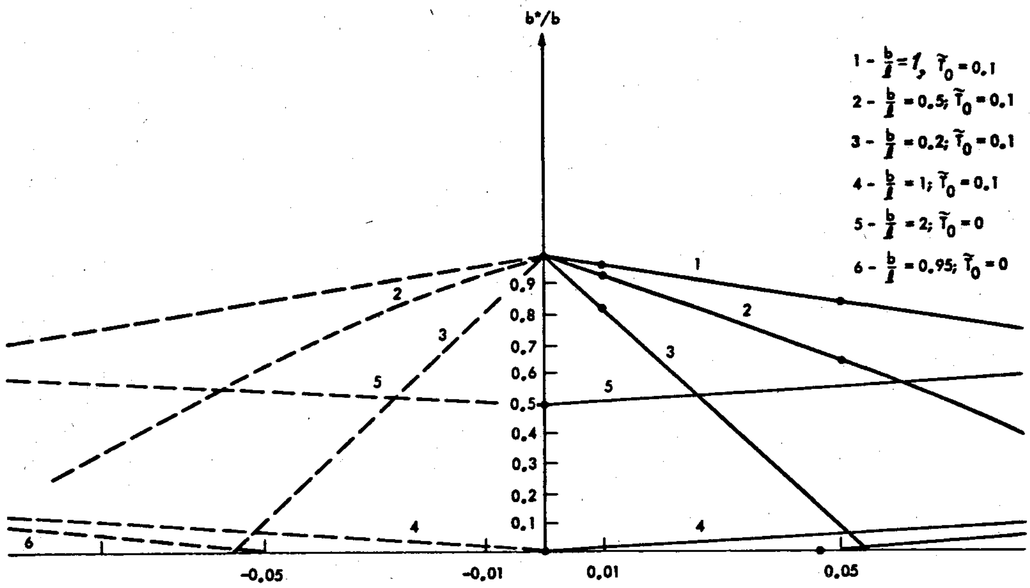

Fig. 5 Dependence $b^*/b = \phi(\alpha, \tilde{T}_0, b/\ell)$.

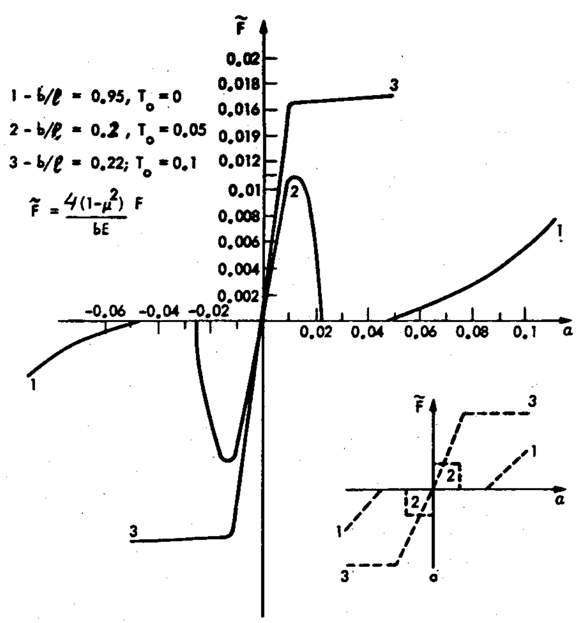

Fig. 6 In-plane force-deflection characteristics for shearing film.

bending to the shearing force of the film [Eq. (16)]

$$F_s = F + C\alpha \tag{17}$$

Nonlinear Transverse In-Plane Vibrations

All of the essential dynamical properties of vibrations with the nonlinearities described above can be successfully investigated by means of the method of equivalent linearization,[6] which will be illustrated for the case $T_0 = 0$ (Fig. 6). For this case Eq. (16) is simplified

$$\tilde{F} = \alpha[1 - (\ell/b)(1-\alpha) + |1 - (\ell/b)(1-\alpha)|] \tag{18}$$

$$F = (bE/4)\tilde{F}$$

Clearly

$$F = 0 \text{ if } 0 \leq \alpha \leq 1 - (b/\ell) \text{ at } (b/\ell) < 1 \tag{19}$$

Thus the zone of insensitivity (see Fig. 6, curve 1) is given by

$$\tau = 1 - (b/\ell) \tag{20}$$

The angle γ is approximated by the following equation

$$\gamma \simeq \mathrm{arc}tg(\ell/b) \tag{21}$$

According to the method of equivalent linearization, the nonlinear vibrations are sought in the form of the first harmonic of the corresponding periodic solution

$$\alpha = \alpha_0 \sin\omega t \tag{22}$$

and Eq. (18) is replaced by

$$\tilde{F} = \tilde{q}(a_0)\alpha \tag{23}$$

where

$$\tilde{q}(a_0) = \frac{1}{\pi a_0}\int_0^{2\pi} \sin\psi\left[1 - \frac{b}{\ell}(1-\sin\psi)\right] + \left[\left|1 - \frac{b}{\ell}(1-\sin\psi)\right|\right]\sin\psi d\psi \simeq \frac{\ell}{b} - \frac{2\ell}{\pi b} \times \left\{\arcsin\frac{1-(b/\ell)}{a_0} + \frac{1-(b/\ell)}{a_0}\sqrt{1 - \frac{[1-(b/\ell)]^2}{a_0^2}}\right\} \tag{24}$$

Now the governing equation of the transverse in-plane vibrations of the film together with the boom in the simplest form is given by

$$m\ddot{a} + [q(a_0) + C]\alpha = 0 \tag{25}$$

where m is the effective mass of the system. Then the frequency of the vibration will depend on its amplitude

$$\omega = \sqrt{\frac{q(a_0) + C}{m}}, \quad q(a_0) = \frac{bE}{4}\tilde{q}(a_0) \tag{26}$$

where $\tilde{q}(a_0)$ is defined by Eq. (24). Obviously, the frequency is increasing with amplitude if $\ell/b > 1$ (Fig. 6). But it is easy to verify that for the other types of nonlinearities (see Fig. 5) the frequency will be decreasing with amplitude.

Induced Out-of-Plane Vibrations

Let us consider out-of-plane vibrations of the solar array film induced by the above-described in-plane vibrations. Restricting attention on the small vibrations, one can use the following governing equation[5]

$$\rho\frac{\partial^2 u}{\partial t^2} = T_{xx}\frac{\partial^2 u}{\partial x^2} + 2T_{xy}\frac{\partial^2 u}{\partial x \partial y} + T_{yy}\frac{\partial^2 u}{\partial y^2} \tag{27}$$

where ρ is the film density and u the out-of-plane small deflections.

In the course of in-plane vibrations the stresses T_{xx}, T_{xy}, and T_{yy} depend on plane coordinates x, y, and time in accordance with Eqs. (4), (8), (9), and (22).

In order to simplify the problem one can assume that the angle β is sufficiently small, i.e.,

$$\sin\beta \sim \beta, \quad \cos\beta \sim 1 \tag{28}$$

which is true for the large prestresses

$$\tilde{T}_0 > \alpha_{\max} \tag{29}$$

Then, according to Eqs. (8-10)

$$T_{xx} \simeq 0$$

$$T_{yy} = \frac{E}{2}\left(\tilde{T}_0^1 + \sqrt{\tilde{T}_0^2 + a_0^2 \sin^2\omega t}\right)$$

$$T_{xy} = 0 \tag{30}$$

and Eq. (27) is simplified to

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{2\rho}\left(\tilde{T}_0^1 + \sqrt{\tilde{T}_0^2 + a_0^2 \sin^2\omega t}\right)\frac{\partial^2 u}{\partial y^2} \tag{31}$$

where ω is defined by Eq. (26), whereas $q(a_0)$ is obtained from the general expression of the equivalent linearization

$$q(a_0) = \frac{1}{\pi a_0}\int_0^{2\pi} F(\sin\psi)\sin\psi \, d\psi \tag{32}$$

Here $F(\alpha)$ is given by Eq. (16).

The solution to Eq. (31) can be expressed in the following form

$$U = T(t)\,Y(y) \tag{33}$$

Separating variables,

$$\ddot{T} + \lambda\frac{E}{\rho}\left(\tilde{T}_0^1 + \sqrt{\tilde{T}_0^2 + a_0^2 \sin^2\omega t}\right)T = 0 \tag{34}$$

$$Y'' + \lambda Y = 0 \tag{35}$$

where the constant λ must be determined from the boundary conditions.

Equation (35) together with the boundary conditions

$$Y(0) = 0, \quad Y(\ell) = 0 \tag{36}$$

leads to the following eigenvalues

$$\lambda_n = (\pi n/\ell)^2, \quad n = 1, 2, \ldots \tag{37}$$

Thus

$$Y_n = \sin(\pi n/\ell)y \tag{38}$$

substituting Eq. (37) into Eq. (34) yields

$$\ddot{T}_n + \left(\frac{\pi n}{\ell}\right)^2\frac{E}{\rho}\left(\tilde{T}_0^1 + \sqrt{\tilde{T}_0^2 + a_0^2 \sin^2\omega t}\right)T_n = 0 \tag{39}$$

Taking Eq. (29) into account one can use the following approximations

$$\sqrt{\tilde{T}_0^2 + a_0^2\sin^2\omega t} \simeq \tilde{T}_0\left(1 + \frac{a_0^2}{2\tilde{T}_0^2}\sin^2\omega t\right)$$

$$\simeq \tilde{T}_0\left[1 + \frac{a_0^2}{2\tilde{T}_0^2}\left(\frac{1}{2} - \frac{1}{2}\cos 2\omega t\right)\right] \tag{40}$$

which simplifies Eq. (39) to

$$\ddot{T}_n + \Omega_n^2(1 - 2\mu_n \cos 2\omega t)T_n = 0 \tag{41}$$

where

$$\Omega_n^2 = \left(\frac{\pi n}{\ell}\right)^2\frac{E}{\rho}\left(\tilde{T}_0^1 + \tilde{T}_0 + \frac{a_0^2}{4\tilde{T}_0}\right)$$

$$\mu_n = \left(\frac{\pi n}{\ell}\right)^2\frac{E}{\rho}\left(\tilde{T}_0^1 + \tilde{T}_0 + \frac{a_0^2}{4\tilde{T}_0}\right) \tag{42}$$

Equation (41) is the well-known Mathieu equation.[9] One of the most interesting characteristics of this equation is that, for certain relationships between its coefficients, it has solutions which are unbounded. Such solutions correspond to a parametric resonance between in-plane and out-of-plane vibrations,

$$\omega = 2\Omega_n \tag{43}$$

In order to describe the situation in this region one should consider the nonlinear coupled system of the governing equations for large in-plane and out-of-plane vibrations.[5] Usually the nonlinear couplings of vibrations stabilize the system, leading to regimes of coupled self-oscillations in which frequencies and amplitudes do not depend on initial conditions.

Conclusions

A nonlinear structural model of a solar array has been developed to explain tests data and to predict new effects in nonlinear in-plane and out-of-plane vibrations. The analytical expression for the shear force-deflection characteristic fits the test data, which previously have not been understood fully from the point of view of linear theory.

Acknowledgments

This paper presents the results of one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under Contract NAS 7-100 sponsored by NASA. The effort was sponsored by Dr. L. Harris, Materials and Structures Division, Office of Aeronautics and Space Technology, National Aeronautics and Space Administration.

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